CHAPTER 7

This Flowing World

"Our first naïve impression of Nature and matter is that of continuity. Be it a piece of metal or a volume of liquid, we invariably conceive it as divisible into infinity, and ever so small a part of it appears to us to possess the same properties as the whole."

—David Hilbert

In mathematics all roads lead back to Greece.

Here I am about to sketch the evolution of the idea of the infinitesimal. The place where the concept matured is Western Europe, and the time the seventeenth and the eighteenth centuries; yet when I endeavor to trace the origin of the idea I see another place and another time: the scene shifts back to classical Greece and to the memorable days of Plato.

The problem of the infinite, like the closely related problem of irrationals, grew up on Greek soil. There also occurred its first crisis, and it has had many since. The crisis came in the days of Plato, but it was not of Plato's making. Nor had the other orthodox philosophers of Greece any claim to having raised the issue. It was precipitated by a school of thinkers whom the leading philosophers of the period contemptuously called the "Sophists."

"Eleates" was the other name by which the orthodox thinkers stamped these obscure men, implying, perhaps, that their teachings were just as outlandish and insignificant as the homeland of their chief representative, Parmenides and Zeno. For Elea was a poor Greek colony in Southern Italy, "possessed of no other importance," says Laërius, "than the knowledge of how to raise virtuous citizens." To us, however, in retrospect, it seems that the Sophists were Elea's only claim to fame.

"The Arguments of Zeno of Elea have, in one form or another," says Russell, "afforded grounds for almost all the theories of space and time and infinity which have been constructed from his day to our own." Yet we don't know today whether these arguments were presented in the course of a debate or whether they appeared in the form of a book. Perhaps both! For we read in Plato's dialogue "Parmenides," one of the few sources we have on the obscure subject, of a visit which Zeno made to Athens in the company of his master, Parmenides. There is reference there to a previous visit during which, it appears, Zeno had presented his arguments. Yet when asked about these, Zeno replies:

> "Zeal for my master led me to write the book in the days of my youth, but one stole the writing; and, therefore, I had no choice whether it should become public; the motive for writing it was not the ambition of an older man, but the pugnacity of a young one."

Be this as it may, we know of the arguments only through Aristotle. Could the Stagyrite have resisted the temptation to distort the arguments of a dead adversary?

The rendition of the arguments in modern language is very difficult. Not that there is a dearth of translations—quite the contrary: we are suffering here from an *embarras du choix*. There are scores of translations and hundreds of paraphrases, and as for interpretations, no obscure passage in the Scriptures has been more honored. Each rendition reflects its author's pet theory, and there are almost as many theories as there are authors. The four Arguments of Zeno as recorded by Aristotle in his *Physica* are:

The First Argument: Dichotomy:

"The first is the one on the non-existence of motion, on the ground that what moves must always attain the middle point sooner than the end point."

The Second Argument: Achilles and the Tortoise:

"The second is the so-called Achilles. It consists in this, that the slower will never be overtaken in its course by the quicker, for the pursuer must always come first to the point from which the pursued has just departed, so the slower must necessarily be always still more or less in advance."

The Third Argument: The Arrow:

"If everything, when it is behaving in a uniform manner, is continually either moving or at rest, but what is moving is always in the now, then the moving arrow is motionless."

The Fourth Argument: The Stadium:

"The fourth is that concerning two rows, each row being composed of an equal number of bodies of equal size, passing each other on a race course, as they proceed with equal velocity in opposite directions; the one row originally occupying the space between the god and the middle point of the course, and the other that between the middle point and the starting point. This, he thinks, involves the conclusion that half a given time is equal to double the time."

Those who are metaphysically inclined see in the Arguments a refutation of the reality of motion. Others, like the historian Tannery, claim that Zeno had no such intention, but that, on the contrary, he used the undisputed reality of motion to point out the flagrant contradictions which reside in our notions of space, time, and continuity. Closely allied to this view is the opinion of Henri Bergson, who maintains that "the contradictions pointed out by the Eleatic school concern much less motion itself than the artificial reorganization of motion performed by our mind."

From this last point of view the value of the Arguments lies precisely in the fact that they forcefully bring out the position which mathematics occupies in the general scheme of human knowledge. The Arguments show that space and time and motion as perceived by our senses (or for this matter by their modern extensions, the scientific instruments) are not co-extensive with the mathematical concepts which bear the same name. The difficulties raised by Zeno are not of the type to alarm the pure mathematician—they do not disclose any logical contradictions, but only sheer ambiguities of language; the mathematician may dispose of these ambiguities by admitting that the symbolic world in which he creates is not identical with the world of his senses.

Thus the alleged properties of the straight line are of the geometer's own making. He deliberately disregards thickness and breadth, deliberately assumes that the thing common to two such lines, their point of intersection, is deprived of all dimension. Desirous of applying the laws of arithmetic to these geometrical beings, he admits, as we shall see, the validity of infinite processes, of which the infinte divisibility of a segment, the *dichotomy* of the Greeks, is but a particular instance. Classical geometry is a logical consequence of these assumptions, but the assumptions themselves are abitrary, a convenient fiction at best. The mathematician could reject the classical postulates, one or all, and substitute for them a new body of assumptions; he could, for instance, take for new elements the *stripe* and the *area* common to two stripes, and, calling these elements lines and points, construct a geometry altogether different from the classical doctrine, but just as consistent and perhaps just as fruitful.

But to the practical man, to the physicist, to the engineer, not all such systems are equally acceptable. The practical man demands an appearance of reality at least. Always dealing in the concrete, he regards mathematical terms not as symbols or thought but as images of reality. A system acceptable to the mathematician because of its inner consistency may appear to the practical man to be full of "contradictions" because of the incomplete manner in which it represents reality.

Strange though it may seem, it is the practical man who should be deeply concerned with the Arguments, because they attack the validity of the application of mathematics to physical reality. But, happily enough, the practical man is rarely interested in arguments.

The historical importance of the Arguments cannot be overestimated. For one thing, they forced the Greeks to adopt a new attitude towards the concept of time.

What Zeno substantially says in his first argument is this: The runner before reaching his goal must reach the midpoint of the course, and it takes him a *finite* time to achieve this. He also must reach the midpoint of the remaining distance, and this too will take a *finite* time. Now *what has been said once can always be repeated.* There are an infinite number of stages in his traversing of the race-course, and each one of these stages requires a finite time. But the sum of an infinite number of finite intervals is infinite. The runner will therefore never attain his goal.

Aristotle disposes of this argument as follows:

> "Time and space are divided into the same and equal divisions. Wherefore also, Zeno's argument, that it is impossible to go through an infinite collection or to touch an infinite collection one by one in a finite time, is fallacious. For there are two senses in which the term 'infinte' is applied both to length and to time and in fact to all continuous things: either in regard to divisibility or in regard to number. Now it is not possible to touch things infinite as to number in a finite time, but it is possible to touch things infinite in regard to divisibility; for time itself is also infinite in this sense."

Thus the net result of the first two arguments (for the second is just an ingenious paraphrase of the first) is that it is impossible to assume *dichotomy of space* without simultaneously admitting *dichotomy of time.* But this is precisely what it is so difficult to grasp! For the divisibility of a line is easily conceived: we can readily materialize it by cutting a stick or marking a line. But "marking time" is just a figure of speech: time is the one thing on which we cannot experiment: it is either all in the past or all in the future. Dividing time into intervals was just an act of the mind to the Greeks, and is just an act of the mind to us.

Endowing time with the attribute of infinite divisibility is equivalent to representing time as a geometrical line, to identifying *duration* with *extension.* It is the first step towards the *geometrization* of mechanics. Thus the first arguments of Zeno were directed against the principle on which the four-dimensional world of modern Relativity is built.

The real punch of the Arguments was reserved for the last two; as though Zeno foresaw the defense of his opponents and prepared to meet it. The fourth, which contains the germ of the problem of Relativity, does not concern us here. It is the third argument that forcefully exposes the chasm between motion as perceived by our senses and the mathematical fiction which masquerades under the same name.

We can hear Zeno's answer in rebuttal:

> "You say that just as space consists of an infinity of contiguous points, so time is but an infinite collection of contiguous instants? Good! Consider, then, an arrow in its flight. At any instant its extremity occupies a definite point in its path. Now, while *occupying this position* it must be *at rest* there. But how can a point be motionless and yet in motion at the same time?"

The mathematician disposes of this argument by *fiat:* Motion? Why, motion is just a correspondence between position and time. Such a correspondence between variables he calls a *function.* The law of motion is just a function, in fact the prototype of all *continuous functions.* Not different in substance from the case of a cylinder filled with gas and provided with a piston which is free to slide within the cylinder. To every possible position of the piston there will correspond a definite pressure within the cylinder. To obtain the pressure corresponding to any position we may stop the piston in this position and read the pressure gauge.

But is it the same with a moving body? Can we stop it at any instant without curtailing the very motion which we are observing? Assuredly not! What is it then that we mean by the moving body's *occupying a certain position at a certain time?* We mean that while we cannot conceive of a physical procedure which will arrest the arrow in its flight without destroying the flight, there is nothing to prevent our doing so by an *act of the mind.* But the only reality behind this act of mind is that *another* arrow can be imagined as motionless at this point and at this instant.

Mathematical motion is just an infinite succession of states of rest, i.e., mathematics reduces dynamics to a branch of statics.

The principle that accomplishes this transition was first formulated by d'Alembert in the eighteenth century. This identification of motion with a succession of contiguous states of rest, during which the moving body is in equilibrium, seems absurd on the face of it. And yet motion made up of *motionless* states is no more, nor less absurd than length made up of *extensionless* points, or time made up of *durationless* instants.

True, this abstraction is not even the skeleton of the real motion as perceived by our senses! When we see a ball in flight we perceive the motion as a whole and not as a succession of infinitesimal jumps. But neither is a mathematical line the true, or even the fair, representation of a wire. Man has for so long been trained in using these fictions that he has come to prefer the substitute to the genuine article.

The subsequent course of Greek science shows clearly how great was the influence which the crisis precipitated by the Arguments of Zeno exercised on the mathematical thought of the Hellenes.

On the one hand this crisis ushered in an era of sophistication. It was the natural reaction from the naïve verbiage of the Pythagoreans, that strange mixture of mathematical ideas with religious slogans and vague metaphysical speculations. What a contrast to this is the sever rigor of Euclid's *Elements*, which to this day serves as a model for mathematical disciplines!

On the other hand, by instilling into the mind of the Greek geometers the *horror infiniti*, the Arguments had the effect of a partial paralysis of their creative imagination. The infinite was taboo, it had to be kept out, at any cost; or, failing in this, camouflaged by arguments *ad absurdum* and the like. Under such circumstances not only was a positive theory of the infinite impossible, but even the development of infinite processes, which had reached quite an advanced stage in pre-Platonic times, was almost completely arrested.

We find in classical Greece a confluence of most fortunate circumstances: a line of geniuses of the first rank, Eudoxus, Aristarchus, Euclid, Archimedes, Apollonius, Diophantus, Pappus; a body of traditions which encouraged creative effort and speculative thought and at the same time furthered a critical spirit, safeguarding the investigator against the pitfalls of an ambitious imagination; and finally, a social structure most propitious to the development of a leisure class, providing a constant flow of thinkers, who could devote themselves to the pursuit of ideas without regard to immediate utility—a combination of circumstances, indeed, which is not excelled even in our own days. Yet Greek mathematics stopped short of an algebra in spite of a Diophantus, stopped short of an analytic geometry in spite of an Apollonius, stopped short of an infinitesimal analysis in spite of an Archimedes. I have already pointed out how the absence of a notational symbolism thwarted the growth of Greek mathematics; the *horror infiniti* was just as great a deterrent.

In the *method of exhaustion,* Archimedes possessed all the elements essential to an infinitesimal analysis. For modern analysis is but the theory of infinite processes, and infinite processes have for foundation the idea of *limit.* The precise formulation of this idea I reserve for the next chapter. It is sufficient to say here that the idea of limit as conceived by Archimedes was adequate for the development of the calculus of Newton and Leibnitz and that it remained practically unchanged until the days of Weierstrass and Cantor. Indeed the *calculus of limits* rests on the notion that two variable magnitudes will approach a state of equality if their difference could be made deliberately small, and this very idea is also the basis of the method of exhaustion.

Furthermore, the principle provides an actual method for determining the limit. This consists in "trapping" the variable magnitude between two others, as between the two jaws of a vise. Thus in the case of the periphery of the circle, of which I have already spoken, Archimedes grips the circumference between two sets of regular polygons of an increasing number of sides, of which one set is circumscribed to the circle and the other is inscribed in it. As I said before, Archimedes showed by this method that the number π is contained between $3\frac{1}{7}$ and $3\frac{10}{71}$. By this method he also found that the area under a parabolic arch is equivalent to two-thirds of the area of a rectangle of the same base and altitude—the problem which was the precursor of our modern integral calculus.

Yes, in all justice it must be said that Archimedes was the founder of infinitesimal analysis. What the method of exhaustion lacked of being the integral calculus of the eighteenth century was a proper symbolism, and a positive—or, shall I say, a naïve—attitude towards the infinite. Yet no Greek followed in the footsteps of Archimedes, and it was left to another epoch to explore the rich territory discovered by the great master.

When, after a thousand-year stupor, European thought shook off the effect of the sleeping powders so skillfully administered by the Christian Fathers, the problem of infinity was one of the first to be revived.

Characteristic of this revival, however, was the complete absence of the critical rigor of the Greeks, and this in spite of the fact that Renaissance mathematics relied almost entirely on Greek sources. The rough-and-ready methods inaugurated by

Kepler and Cavalieri were continued, with only a pretense of refinement, by Newton and Leibnitz, by Wallis, the inventor of the symbol for infinity, by the four Bernoullis, by Euler, by d'Alembert.

They dealt with infinitesimals as fixed or variable according to the exigencies of the argument; they manipulated infinite sequences without much rhyme or reason; they juggled with limits; they treated divergent series as if these obeyed all rules of convergence. They defined their terms vaguely and used their methods loosely, and the logic of their arguments was made to fit the dictates of their intuition. In short, they broke all the laws of rigor and of mathematical decorum.

The veritable orgy which followed the introduction of the infinitesimals, or the *indivisibilia,* as they were called in those days, was but a natural reaction. Intuition had too long been held imprisoned by the severe rigor of the Greeks. Now it broke loose, and there were no Euclids to keep its romantic flight in check.

Yet another cause may be discerned. It should be remembered that the brilliant minds of that period were all raised on scholastic doctrine. "Let us have a child up to the age of eight," said a Jesuit once, "and his future will take care of itself." Kepler reluctantly engaged in astronomy after his hopes of becoming an ecclesiastic were frustrated; Pascal gave up mathematics to become a religious recluse; Descartes's sympathy for Galileo was tempered by his faith in the authority of the Church; Newton in the intervals between his masterpieces wrote tracts on theology; Leibnitz was dreaming of number schemes which would make the world safe for Christianity. To minds whose logic was fed on such speculations as Sacrament and Atonement, Trinity and Trans-substantiation, the validity of infinite processes was a small matter indeed.

This may be taken as a rather belated retort to Bishop Berkeley. A quarter of a century after the publication of Newton's epoch-making work on the infinitesimal calculus, the bishop wrote a tract entitled: "The Analyst; a Discourse Addressed to an Infidel Mathematician." The contention that too much is taken on faith in matters of religion, the bishop counters by pointing out that the premises of mathematics rest on no securer foundation. With inimitable skill and wit he subjects the doctrine of infinitesimals to a searching analysis and discloses a number of loose arguments, vague statements, and glaring contradictions. Among these are the terms "fluxion" and "difference"; and against these the bishop directs the shafts of his splendid Irish humor: "He who can digest a second or third fluxion, a second or third difference, need not, methinks, be squeamish about any point in Divinity."

The "fluxions" of Newton, the "differences" of Leibnitz, are today called *derivatives* and *differentials.* They are the principal concepts of a mathematical discipline which, together with analytical geometry, has grown to be a powerful factor in the development of the applied sciences: the *Differential and Integral Calculus.* Descartes is credited with the creation of analytic geometry; the controversy as to whether it was Newton or Leibnitz who first conceived the calculus raged throughout the eighteenth century and is not quite settled even today. And yet, we find the principles of both disciplines clearly indicated in a letter which Fermat addressed to Roberval, dated October 22, 1636, a year before Descartes's *Geometria* appeared, and sixty-eight years before the publication of Newton's *Principia.* If it were not for Fermat's unaccountable habit of not publishing his researches, the creation of both analytic geometry and the calculus would have been credited to this Archimedes of the Renaissance, and the mathematical world would have been spared the humiliation of a century of nasty controversy.

The substance of Newton's principle can be illustrated by the example of motion, which, incidentally, was the first subject to which the differential calculus was applied. Consider a particle in motion along a straight line. If in equal times equal spaces are covered, then the particle is said to move *uniformly;* and the distance covered in a unit of time, say a second, is called the *velocity* of this uniform motion. Now if the distances covered in equal intervals of time are not equal, i.e., if the motion is *non-uniform,* there is no such thing as velocity in the sense in which we have just used the word. Yet we may divide the distance which was covered in a certain interval by the time interval and call this ratio the *average* velocity of the particle in this interval. Now it is this ratio that Newton would call *prime* ratio. This number, however, obviously depends on the length of the interval considered. However, notice that the smaller the interval the closer does the velocity approach a certain fixed value. … We have here an example of *a sequence* in which the difference between succeeding terms is growing continually less until after a while two contiguous terms will become indistinguishable. Now let us conceive (and such a conception is justified by our intuitive notion of the *continuity of space and time)* that we continue diminishing the interval of time *indefinitely.* Then, the ultra-ultimate term of the sequence (the *ultima ratio* of Newton) will, according to Newton, represent the *velocity at the point at the beginning of the interval.*

Today we say: *by definition* the velociy of the moving point at any time is the limiting value of the average velocity when the interval to which the average velocity pertains diminishes indefinitely. In the days of Newton they were not so careful.

The ultimate ratios were also called by Newton *fluxions.* The fluxion was the rate of change of a variable magnitude, such as length, area, volume, pressure, etc. These latter Newton called the *fluents.* It is to be regretted that these expressive words were not retained, but were replaced by such indifferent terms as *derivative* and *function.* For the Latin *fluere* means "to flow"; *fluent* is "the flowing," and *fluxion* "the rate of flow."

Newton's theory dealt with continuous magnitudes and yet postulated the infinite divisibility of space and time; it spoke of a flow and yet dealt with this flow as if it were a succession of minute jumps. Because of this, the theory of fluxions was open to all the objections that two thousand years before had been raised by Zeno. And so the age-long feud between the "realists," who wanted a mathematics to comply with the crude reality of man's senses, and the "idealists," who insisted that reality must conform to the dictates of the human mind, was ready to be resumed. It only awaited a Zeno, and the Zeno appeared in the strange form of an Anglican ecclesiastic. But let me leave the word to George Berkeley, later Bishop of Cloyne:

> "Now, as our Sense is strained and puzzled with the perception of objects extremely minute, even so the Imagination, which faculty derives from Sense, is very much strained and puzzled to frame clear ideas of the least particles of time, or the least increments generated therein; and much more so to comprehend the moments, or those increments of the flowing quantities *in statu nascenti,* in their very first origin or beginning to exist, before they become finite particles. And it still seems more difficult to conceive the abstract velocities of such nascent imperfect entities. But

the velocities of the velocities—the second, third, fourth, and fifth velocities, etc.—exceed, if I mistake not, all human understanding. The further the mind analyseth and pursueth these fugitive ideas the more it is lost and bewildered; the objects, at first fleeting and minute, soon vanishing out of sight. Certainly, in any sense, a second or a third fluxion seems an obscure Mystery. The incipient celerity of an incipient celerity, the nascent augment of a nascent augment, i.e., of a thing which hath no magnitude—take it in what light you please, the clear conception of it will, if I mistake not, be found impossible ...

"The great author of the method of fluxions felt this difficulty, and therefore he gave in to those nice abstractions and geometrical metaphysics without which he saw nothing could be done on the received principles. ...It must, indeed, be acknowledged that he used fluxions like the scaffold of a building, as things to be laid aside or got rid of as soon as finite lines were found proportional to them. But then these finite exponents are found by the help of fluxions....And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities? ...

"And, to the end that you may more clearly comprehend the force and design of the foregoing remarks, and pursue them still farther in your own meditations, I shall subjoin the following Queries. ...

"*Query 64.* Whether mathematicians, who are so delicate in religious points, are strictly scrupulous in their own science? Whether they do not submit to authority, take things upon trust, and believe points inconceivable? Whether they have not their mysteries, and what is more, their repugnances and contradictions?"

And the net result of Berkeley's witty perorations? Well, in so far as it attacked inaptness and inconsistency in the mathematical terminology, it performed a genuine service. Succeeding decades saw a considerable change: such words as prime and ultimate, nascent and incipient, fluent and fluxion, were abandoned. The *indivisibilia* became the *infinitesimals* of today; the infinitesimal being merely a variable quantity that approaches zero as a limit. The whole situation became slowly but surely dominated by the central idea of limit.

Had Bishop Berkeley reappeared fifty years after he wrote "The Analyst" he would not have recognized the child he had scolded, so modest had it become. But would he have been satisfied? Not Berkeley! For the sharp eyes of the acute bishop would have detected the same leopard behind the changed spots. What he had objected to was not so much the lack of conciseness in language (although this too came in for its share in his critique); but rather what Zeno had pointed out: the failure of the new method to satisfy our intuitive idea of the continuous as of something uninterrupted, something indivisible, something that had no parts, because any attempt to sever it into parts would result in the destruction of the very property under analysis.

And if we strain our imaginations still more and imagine the bishop re-appearing in our own midst, we would hear him raising the same objections, leveling the same accusations. But this time to his surprise and delight he would find in the enemy camp a powerful party of men who would not only defend him but hail him as a pioneer.

But of this later.

And in the meantime analysis grew and grew, not heeding the warnings of the critics, constantly forging ahead and conquering new domains. First geometry and mechanics, then optics and acoustics, propagation of heat and thermodynamics, electricity and magnetism, and finally even the laws of the Chaos came under its direct sway.

Says Laplace:

> "We may conceive the present state of the universe as the effect of its past and the cause of its future. An Intellect who at any given instant knew all the forces that animate nature and the mutual position of the beings who compose it, were this Intellect but vast enough to submit his data to analysis, could condense into a single formula the movement of the greatest body in the universe and that of the lightest atom; to such an Intellect nothing would be uncertain, for the future, even as the past, would be ever present before his eyes."

And yet this magnificent structure was created by the mathematicians of the last few centuries without much thought as to the foundations on which it rested. Is it not remarkable then, that in spite of all the loose reasoning, all the vague notions and unwarranted generalization, so few serious errors had been committed? "Go ahead, faith will follow" were the encouraging words with which d'Alembert kept reinforcing the courage of the doubters. As though heeding his words, they did forge ahead, guided in their wanderings by a sort of implicit faith in the validity of infinite processes.

Then came the critical period: Abel and Jacobi, Gauss, Cauchy and Weierstrass, and finally Dedekind and Cantor, subjected the whole structure to a searching analysis, eliminating the vague and ambiguous. And what was the net result of this reconstruction? Well, *it condemned the logic of the pioneers, but vindicated their faith.*

The importance of infinite processes for the practical exigencies of technical life can hardly be overemphasized. Practically all applications of arithmetic to geometry, mechanics, physics and even statistics involve these processes directly or indirectly.

Indirectly because of the generous use these sciences make of irrationals and transcendentals; directly because the most fundamental concepts used in these sciences could not be defined with any conciseness without these processes. Banish the infinite process, and mathematics pure and applied is reduced to the state in which it was known to the pre-Pythagoreans.

Our notion of the length of an arc of a curve may serve as an illustration. The physical concept rests on that of a bent wire. We imagine that we have *straightened* the wire without *stretching* it; then the segment of the straight line will serve as the measure of the length of the arc. Now what do we mean by "without stretching"? We mean without a change in length. But this term implies that we already know something about the length of the arc. Such a formulation is obviously a *petitio principii* and could not serve as a mathematical definition.

The alternative is to inscribe in the arc a sequence of rectilinear contours of an increasing number of sides. The sequence of these contours approaches a limit, and the length of the arc is defined as the limit of this sequence.

And what is true of the notion of length is true of areas, volumes, masses, moments, pressures, forces, stresses and strains, velocities, accelerations, etc., etc. All these notions were born in a *"linear," "rational"* world where nothing takes place but what is straight, flat, and uniform. Either, then, we must abandon these elementary rational notions—and this would mean a veritable revolution, so deeply are these concepts rooted in our minds; or we must adapt those rational notions to a world which is neither flat, nor straight, nor uniform.

But how can the flat and the straight and the uniform be adapted to its very opposite, the skew and the curved and the non-uniform? Not by a finite number of steps, certainly! The miracle can be accomplished only by that miracle-maker the *infinite.* Having determined to cling to the elementary rational notions, we have no other alternative than to regard the "curved" reality of our senses as the ultra-ultimate step in an infinite sequence of *flat* worlds which exist only in our imagination.

The miracle is that it works!