CHAPTER 11

The Anatomy of the Infinite

"The essence of mathematics is its freedom."
—Georg Cantor

To measure the *plurality* of an infinite collection seems a bizarre idea at first. Yet even those least conversant with mathematical ideas have a vague feeling that there are infinities *and* infinities—that the term infinity as attached to the natural sequence of numbers and the term infinity used in connection with points on a line are essentially different.

This vague idea which we have the "contents" of an infinite collection may be compared to a net. It is as if we threw a net of unit mesh and so singled out the whole numbers, all other numbers passing through. We then took a second net of mesh 1/10; then a third of mesh 1/100; and continuing this way we gathered up more and more of the rational numbers. We conceive of no limit to the refinement of this process, for no matter how fine a net we may cast, there is yet a finer that *could be* cast. Give free rein to your imagination and you can picture this ultra-ultimate net so compact, of mesh so fine, as to be able to gather up *all* the rational numbers.

It is when we push this analogy to the extreme and begin to regard this limiting net as something fixed, frozen as it were, that we strike all the difficulties so skilfully brought out by Zeno. Here, however, we are concerned with another difficulty.

This ultra-ultimate rational net, even if it could be materialized, would still be incapable of gathering *all numbers.* A net still more "compact" is necessary to take care of the irrationals of algebra; and even this "algebraic" net would be incapable of gathering the transcendental numbers. And thus our intuitive idea is that the rational number domain is more compact than the natural; that the algebraic numbers are arranged in still denser formation; and that finally the real number domain, the arithmetic continuum, is the *ultra-dense medium,* a medium without gaps, a network of mesh zero.

If then we are told for the first time that Georg Cantor made an actual attempt to classify infinite collections and to endow each with a number representative of its plurality, we naturally anticipate that he succeeded in finding a measure of this variable compactness.

And just because such were our anticipations, the achievement of Cantor has many surprises in store for us, some of these so striking as to border on the absurd.

The attempt to measure the compactness of a collection by means of nets is doomed to failure because it is physical in principle, and not arithmetical. It is not arithmetical, because it is not built on the principle of correspondence on which all arithmetic rests. The classification of the *actually infinite,* i.e., of the various types of plurality of infinite collections, if such a classification is at all possible, must proceed on the lines along which pluralities of finite collections were classified.

Now we saw in the opening chapter that the notion of *absolute plurality* is not an inherent faculty of the human mind. The genesis of the natural number, or rather of the cardinal numbers, can be traced to our matching faculty, which permits

us to establish correspondence between collections. The notion of *equal-greater-less* precedes the number concept. We learn *to compare* before we learn *to evaluate.* Arithmetic does not begin with numbers; it begins with *criteria.* Having learned to apply these criteria of equal-greater-less, man's next step was to devise models for each *type* of plurality. These models are deposited in his memory very much as the standard meter is deposited at the Bureau of Longitudes in Paris. One, two, three, four, five ...; we could just as well have had: I, wings, clover, legs, hand ... and, for all we know, the latter preceded our present form.

The principle of correspondence generates the integer and through the integer dominates all arithmetic. And, in the same way, before we can measure the plurality of infinite collections, we must learn to compare them. How? In the same way that this was accomplished for finite collections. The matching process, which performed such signal services in finite arithmetic, should be extended to the *arithmetic of the infinite:* for the elements of two infinite collections might also be matched one by one.

The possibility of establishing a correspondence between two infinite collections is brought out in one of the dialogues of Galileo, the first historical document on the subject of infinite aggregates. I reproduce this dialogue verbatim from a book entitled "Dialogs Concerning the New Sciences," which appeared in 1636. Three persons participate in the dialogues. Of these Sagredo represents the practical mind, Simplicio one who is trained in scholastic methods, while Salviati is obviously Galileo in person.

> *Salviati:* This is one of the difficulties which arise when we attempt with our finite minds to discuss the infinite, assigning to it those properties which we give to the finite and limited; but

this I think is wrong for we cannot speak of infinite quantities as being the one greater or less than or equal to the other. To prove this, I have in mind an argument which for the sake of clearness, I shall put in the form of questions to Simplicio, who raised this difficulty.

I take it for granted that you know which of the numbers are squares and which are not.

Simplicio: I am quite aware that a square number is one which results from the multiplication of another number by itself; thus, 4, 9, etc., are square numbers which come from multiplying 2, 3, etc., by themselves.

Salviati: Very well; and you also know that just as the products are called squares, so the factors are called sides or roots; while on the other hand those numbers which do not consist of two equal factors are not squares. Therefore, if I assert that all numbers including both squares and non-squares are more than the squares above, I shall speak the truth, shall I not?

Simplicio: Most certainly.

Salviati: If I ask how many squares there are, one might reply truly that there are as many as the corresponding numbers of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.

Simplicio: Precisely so.

Salviati: But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers, because every number is a root of some square. This being granted, we must say that there are as many squares as there are numbers, because they are just as numerous as their roots, and all the numbers are roots. Yet, at the outset, we have said that there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers. Thus up to 100,

> we have ten, i.e., the squares constitute one-tenth of all numbers; up to 10,000 we find only one hundredth part to be squares, and up to a million only one thousandth part; and, yet, on the other hand, in an infinite number, if one could conceive of such a thing, he would be forced to admit that there are as many squares as there are numbers all taken together.
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> *Sagredo:* What, then, must one conclude under such circumstances?
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> *Salviati:* So far as I see, we can only infer that the number of squares is infinite and the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less" are not applicable to infinite, but only to finite quantities.
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> When, therefore, Simplicio introduces several lines of different lengths and asks how is it possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more, or less, or just as many, points as another, but that each line contains an infinite number.

The paradox of Galileo evidently left no impression on his contemporaries. For two hundred years nothing was contributed to the problem. Then in 1820 there appeared a small tract in German by one Bolzano, entitled "The Paradoxes of the Infinite." This, too, attracted little attention; so little indeed, that when fifty years later the theory of aggregates became the topic of the day, few mathematicians knew who the man was.

Today Bolzano's contributions are of a purely historical interest. While it is true that he was the first to broach the question of the *actually infinite* he did not go far enough. Yet, due honor must be given the man for creating the all-important concept of the *power* of an aggregate of which I shall speak shortly.

The modern *theory of aggregates* begins with Georg Cantor. His essay, which laid the foundation of this new branch of mathematics, appeared in 1883 under the title "On Linear Aggregates." This essay was the first to deal with the actually infinite as with a definite mathematical being. The following passage from this essay will bring out clearly Cantor's approach to the problem:

> "It is traditional to regard the infinite as the indefinitely growing or in the closely related form of a convergent sequence, which it acquired during the seventeenth century. As against this I conceive the infinite in the definite form of something consummated, something capable not only of mathematical formulations, but of definition by number. This conception of the infinite is opposed to traditions which have grown dear to me, and it is much against my own will that I have been forced to accept this view. But many years of scientific speculation and trial point to these conclusions as to a logical necessity, and for this reason I am confident that no valid objections could be raised which I would not be in position to meet."

To appreciate the great courage which it required to break so openly with the traditions of the past, we must understand the universal attitude of Cantor's generation towards the *actually infinite*. For this purpose, I quote from a letter of the great Gauss to Schumacher, which, although written in 1831, set the tone of the mathematical world for the next half-century:

> "As to your proof, I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a figure of speech; an abridged form for the statement that limits exist which certain rations may approach as closely as we desire, while other magnitudes may be permitted to grow beyond all bounds ...

> "... No contradictions will arise as long as Finite Man does not mistake the infinite for something fixed, as long as he is not led by an acquired habit of the mind to regard the infinite as something bounded."

Gauss's ideas on the subject were universally shared, and we can imagine what a storm Cantor's open defiance raised in the camp of the orthodox. Not that the actually infinite was not in one guise or another used in the days of Cantor, but that in such matters the traditional attitude was like that of the Southern gentleman with respect to adultery: he would rather commit the act than utter the word in the presence of a lady.

It was fortunate for Cantor that mature reflection had thoroughly steeled him to face the onslaught, because for many years to come he had to bear the struggle alone. And what a struggle! The history of mathematics has not recorded anything equal to it in fury. The story beginnings of the theory of aggregates show that even in such an abstract field as mathematics, human emotions cannot be altogether eliminated.

Cantor begins where Galileo left off. Yes, it is possible to establish a correspondence between two infinite collections, even if one is but a part of the other! For precision, therefore, let us say that two collections, *finite* or *infinite,* are equivalent, or have the same *power,* if they can be matched element for element. If two collections are of *different power,* then the matching process will exhaust one, but there will still remain unmatched elements in the other. In other words, the first may be matched with *a part* of the second, but the second cannot be matched with any part of the first. Under the circumstances, we say that the second aggregate is of a power *greater* than the first.

If (A) and (B) are two *finite* collections, each containing the same number of elements, then obviously they have the same

power; and, conversely, if (A) and (B) are *finite* collections of equal power, they have also the same *cardinal* number. If (A) and (B) are of unequal power, then to the greater power corresponds also the greater cardinal number. *For finite collections, therefore, the concept of power can be identified with that of cardinal number.* Now, since in the *arithmetic of the finite* the term power can be identified with cardinal number, it is natural to inquire whether it is possible to identify the powers of infinite collections with numbers of a higher order, *transfinite* numbers as it were, and by means of this new concept create a *transfinite arithmetic,* an arithmetic of the infinite.

If we proceed along the lines suggested by the beginnings of finite arithmetic, we must seek model-aggregates, each model representative of some typical plurality. Such models are close at hand: the natural sequence, the rational domain, the field of algebraic numbers, the arithmetic continuum—all these infinite collections which have grown so familiar to us through constant use are admirably adapted as standards of comparison. Let us then endow these standard collections with symbols, and have these symbols play the same rôle in a transfinite arithmetic as their counterparts, the finite cardinal numbers, 1, 2, 3 ..., play in the arithmetic of the finite.

These symbols Cantor calls the *transfinite cardinals.* He orders them in a "sequence" of growing power; he defines the operations of addition, multiplication, and potentiation upon these abstract beings; he shows how they combine among themselves and with finite cardinals. In fine, these illusory creatures of Cantor's genius possess so many of the properties of finite magnitudes that it seems altogether proper to confer upon them the title "number." But one all-important property they do not possess, and that is *finitude.* This last statement sounds like a

truism, and yet it is not intended in any spirit of triviality. All the paradoxical propositions which I am about to present derive from the fact that these mathematical beings, which have all the appearances of numbers, are deprived of some of the most rudimentary attributes of common number. One of the most striking consequences of this definition is that a part of a collection is not necessarily less than the whole: it may be equal to it.

The part may have the power of the whole. This sounds more like theology than mathematics. And, indeed, we find this idea being toyed with by many a theologian and near-theologian. In the Sanskrit manuals, where religion is so delightfully intermixed with philosophy and mathematics and sex instruction, such ideas are quite usual. Thus Bhaskarah in speculating on the nature of the number 1/0 states that it is "like the Infinite, Invariable God who suffers no change when old worlds are destroyed or new ones created, when innumerable species of creatures are born or as many perish."

"The part has the power of the whole." Such is the essence of Galileo's paradox. But while Galileo dodged the issue by declaring that "the attributes of equal, greater, and less are not applicable to infinite, but only to finite quantities," Cantor takes the issue as a point of departure for his theory of aggregates.

And Dedekind goes even further: to him it is characteristic of all infinite collections that they possess parts which may be matched with the whole. For purposes of illustration, consider any infinite sequence ordered and labeled accordingly. Now drop any finite number of terms in the beginning and re-label the curtailed sequence. For every term of the second, there was a term in the original sequence of the same rank, and vice versa. The correspondence is, therefore, complete and the two sequences possess the same power; and yet it cannot be denied

that the second is but a part of the first. Such a phenomenon is possible only in infinite collections, for it is characteristic only of *finite* collections that the whole is never equal to a part.

But let us return to the Cantor theory. The symbol *a* will designate the power of the aggregate of natural numbers. Any aggregate which possesses the power *a* will be called *denumerable*. The sequence of perfect squares, used in Galileo's argument, is such a denumerable aggregate. But so is every other sequence, for the mere fact that we can assign a rank to any term shows that there is a perfect correspondence between the sequence and the natural numbers. The even numbers, the odd numbers, any arithmetic progression, any geometric progression, any sequence at all, is denumerable.

What is more, if any such sequence is imagined removed from the domain of natural numbers, the remaining aggregate is still infinite and still denumerable; and this is why there is no hope of reducing the power of a denumerable set by a *thinning-out* process. We may, for instance, remove all even numbers, then all remaining multiples of 3, then all remaining multiples of 5. We may continue this process indefinitely without affecting the power of what remains.

In the language of Cantor, there is *no smaller transfinite number* than the number *a*, which measures the plurality of any denumerable infinite collection.

But if there be no hope of obtaining a smaller transfinite by thinning out the natural sequence, could we not increase the power by a process of filling-in? It would appear, indeed, that the power of the rational domain, which is everywhere dense,

should be greater than that of the natural sequence which is discrete. Here again our intuition leads us array, for Cantor proves to us that the rational aggregate is also denumerable. To prove this, it is only necessary to show that the rational numbers can themselves be arranged in a sequence, by assigning to each rational number a definite rank. This is what Cantor actually does. We can get a general idea of the method by considering it geometrically.

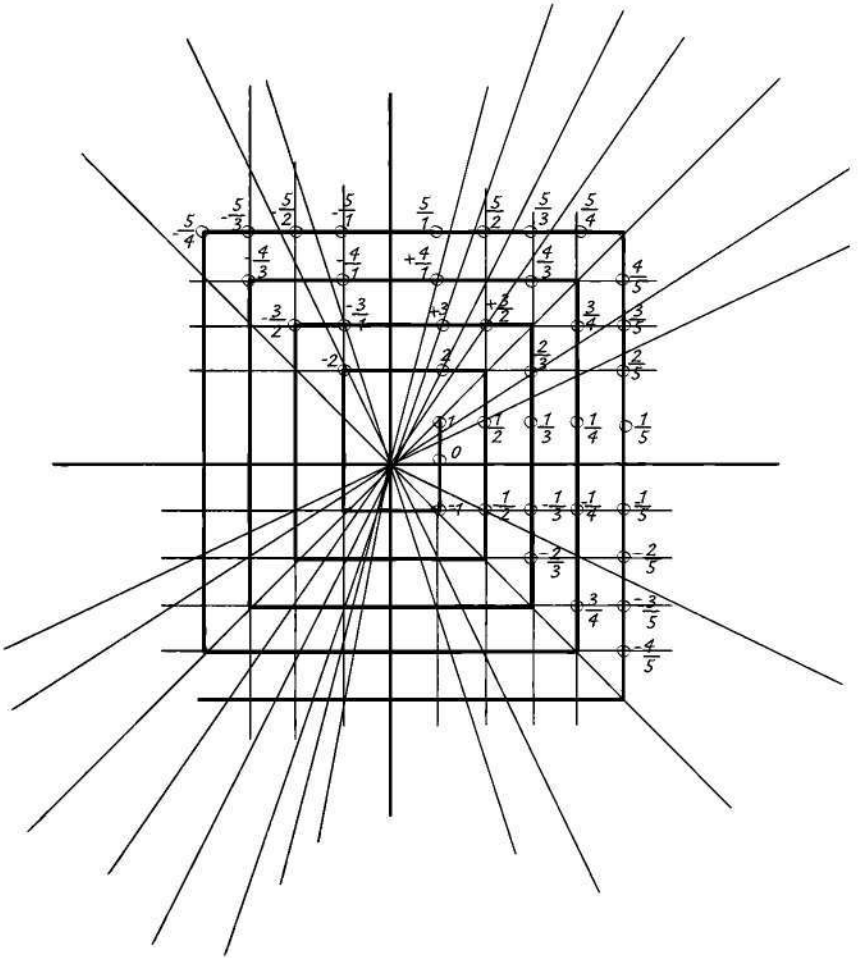
In the accompanying figure we have two sets of parallel lines, at right angles to each other. We identify any line of the horizontal set by a whole number y, the number y taking on all integral values from $-\infty$ to $+\infty$; similarly for the number x, which identifies the vertical lines. Now we label any joint of the infinite lattice we have thus erected by the two numbers which identify the vertical and the horizontal lines there intersecting. Thus the symbol (y, x) identifies a determinate joint in our lattice work, and conversely any point is capable of such a representation.

We shall show that the totality of these joints form a denumerable aggregate. To prove this striking fact, it is sufficient to draw the spiral-like polygonal line as in the figure and follow the joints in the order in which they appear on the diagram.

On the other hand, we can identify the symbol (y, x) with the fraction y/x. But if we do this, it is obvious that we cannot label *all* our joints with distinct rational numbers. In fact, all joints situated on the same line through the origin represent one and the same rational number, as it is easy to convince oneself. To eliminate this ambiguity, *we agree to count each fraction only the first time it occurs.* These points form the sequence:

$$1, 0, -1; -2, 2, +\tfrac{1}{2}, -\tfrac{1}{2}; -\tfrac{3}{2}, -3, +3, +\tfrac{3}{2}, +\tfrac{2}{3}, +\tfrac{1}{3}, \ldots$$

(See figure, page 226.)



DOCUMENTING THE RATIONAL DOMAIN

Here all rationals are represented, and each rational occurs in the sequence but once. *The domain of rational numbers is therefore denumerable.*

But, the reader may exclaim, this stands in direct contradiction to our notion of compactness, according to which no rational number has a successor. Between any two rational numbers we can insert an infinity of others; but here we have actually established a succession! The answer to this is that, while we have here a true succession, it is not of the same type as the natural succession 1, 2, 3, ... which is arranged according to magnitude. We succeeded in enumerating the rational numbers because in the new arrangement we were not obliged to preserve the order of magnitude. We have obtained *succession at the expense of continuity.*

We see that it is essential to discriminate between two kinds of equivalence. From the standpoint of *correspondence,* two collections are equivalent if they can be matched element by element. From the standpoint of *order,* this is also indispensable. But for complete equivalence, for *similarity,* it is necessary in addition that the matching process should not destroy the order of arrangement: i.e., if in the collection (A) the element a preceded the element a', then in the collection (B) the corresponding element b must precede b'. The aggregate of rational numbers arranged according to magnitude, and the spirally arranged aggregate by means of which we denumerated the rational numbers, are equivalent from the standpoint of correspondence, but not from the standpoint of order. In other words, they have the same cardinal number, a, but are of different *ordinal types.*

Hence Cantor proposed a theory of *ordinal types* which form the counterpart of the ordinal numbers of finite arithmetic. Theorem, however, we had the fundamental property that any two collections with equal cardinal numbers had also the

same ordinal number, and to this we owed the facility with which we passed from one to the other. But in the Cantor arithmetic of the infinite, two aggregates may be measured by the same cardinal number and yet be *ordinally distinct,* or, as Cantor says, *dissimilar.*

Thus mere compactness is no obstacle to denumeration, and the filling-in process does not affect the power of an aggregate any more than the thinning-out process did. So the next Cantor deduction is somewhat less of a shock to us; this states that *the aggregate of algebraic numbers is also denumerable.* Cantor's proof of this theorem is a triumph of human ingenuity.

He commences by defining what he calls the *height* of an equation. This is the sum of the *absolute values* of the coefficients of the equation, to which is added its degree diminished by 1. For instance, the equation $2x^3 - 3x^2 + 4x - 5 = 0$ has a height $h = 16$, because $2 + 3 + 4 + 5 + (3 - 1) = 16$.

He proves next that there are but a *finite* number of equations which admit any positive integer h for height. This permits us to order *all* algebraic equations in groups of increasing height; it can be shown that there is only one equation of height 1; three of height 2; twenty-two of height 3, etc.

Now, within each group of given height, we can order the equations by any number of schemes. For instance, we can combine all equations of the same degree into one sub-group and arrange each sub-group according to the magnitude of the first coefficients, for those which have the same first coefficient according to the second, etc., etc.

Any such scheme would allow us to order all algebraic equations into a hierarchy and thus enumerate them, i.e., assign to each equation a rank. Now to every one of these equations may

correspond one or more real roots, but the number is always finite and in fact cannot exceed the degree of the equation—and therefore cannot exceed the height; these roots can again be arranged according to their magnitude. Now if we consider the scheme as a whole, we shall surely find repetitions, but as in the case of the rational numbers, these repetitions can be avoided by agreeing to count any algebraic number only the first time it occurs in the process.

In this manner we have succeeded in assigning to any algebraic number a rank in the hierarchy, or in other words, we have *denumerated the aggregate of algebraic numbers.*

By this time the suspicion will have grown upon the reader that perhaps *all aggregates are denumerable.* If such were the case there would be but one transfinite, and what was true for the rational and the algebraic aggregates would be generally true even of the continuum. By some artifice, such as Cantor's height, any infinite collection could be arranged into a hierarchy and thus enumerated. Such, indeed, was Cantor's idea in the early stages of his work: *to enumerate the real numbers* was one of the points of his ambitious program; and the theory of transfinite numbers owes its birth to this attempt to "count the continuum."

That such is not the case that it is not possible to arrange all real numbers in a denumerable sequence, was known to Cantor as early as 1874. However, the proof of it did not appear until 1883. I cannot go into the details of this demonstration, the general principle of which consists in assuming that all real numbers have been erected into a hierarchy, and then in showing, by a method which we now call the *diagonal procedure,* that it is possible to exhibit other numbers which while real are not among those which have been enumerated.

One sidelight on this proof has an important historical bearing. The reader will remember Liouville's discovery of transcendentals. This existence theorem of Liouville was re-established by Cantor as a sort of by-product of his theorem that the continuum cannot be denumerated. The relative wealth of the two domains, the algebraic and the transcendental, a question which to Liouville had only a vague meaning, was now formulated by Cantor in full rigor. He showed that whereas the algebraic domain has the power a of the aggregate of natural numbers, the transcendentals possess the *power c of the continuum.* Thus the contention that there are incomparably *more* transcendentals than algebraic numbers acquires a true significance.

And here, too, in this domain of real numbers the part may have the power of the whole; in the quaint language of Galileo "the longer line contains no more points than the shorter." In fact, a segment of a line, no matter how short, has the same power as the line indefinitely extended, an area no matter how small has the power of the infinite space of three dimensions. In short: *parceling or piecing together can no more affect the power of an aggregate than thinning out or filling in.*

At this point our intuition again whispers a suggestion. How about these manifolds of higher dimensions: the complex number domain, which we identified with the set of points in the plane; the points in space; the vectors and quaternions; the tensors and the matrices, and other intricate complexes which mathematicians manipulate as though they were individuals, subject to the laws of operations on numbers, but which cannot be represented in a continuous manner as points on a line? Surely these manifolds should have a power higher than that of the *linear* continuum! Surely there are more points in space of

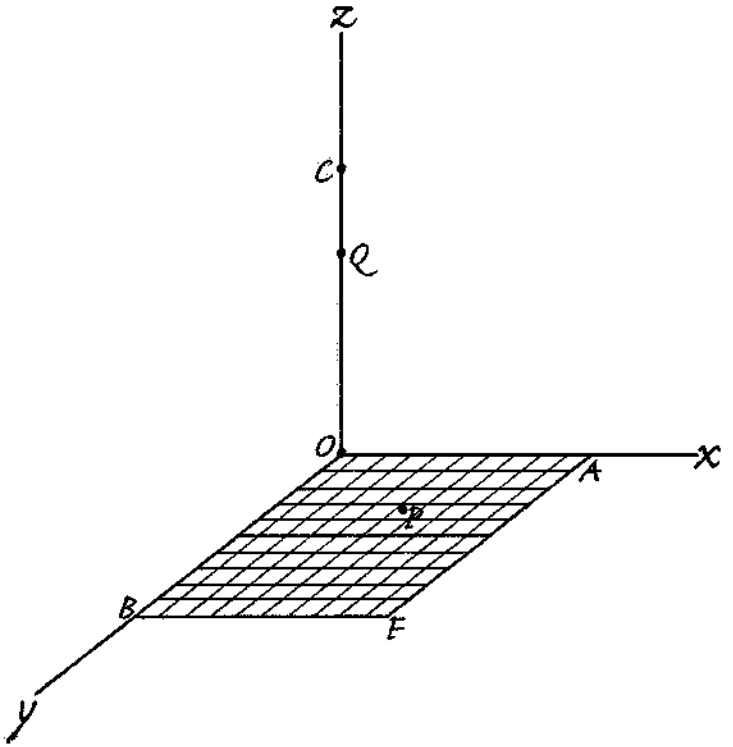
three dimensions, this universe extending indefinitely in all directions, than on a segment of a line one inch long!

This, too, may have been an early idea of Cantor. But he proved conclusively that here too our intuition leads us astray. The infinite manifold of two or three dimensions, the mathematical beings which depend on a number of variables greater even than three, any number in fact, still have no greater power than the *linear continuum.* Nay, even could we conceive of a variable being whose state at any instant depended on an infinite number of independent variables, a being which "lived" in a *world of a denumerable infinite of dimensions,* the totality of such beings would still have a power not greater than that of the linear continuum not greater than a segment one inch long.

This statement strikes us as being in such direct contradiction with our ideas of dimension as to be absurd. Such was, indeed, the opinion of many when Cantor first announced it, and there were first-class minds who took it warily, to say the least. But Cantor's proof of this fundamental proposition is so simple that even a bright child can see it.

I shall illustrate the statement as it applies to the points in the plane: the reader will see that the argument is perfectly general. Since the points within a segment of length 1 have the same power as the indefinite line, and the points within the square of side 1 the same power as the indefinite plane, it will be sufficient to show that a one-to-one correspondence can be established between this square and this segment.

Now any point P within this square OAFB of the accompanying figure can, as we saw, be represented by means of two coordinates x, y. These latter are real numbers not exceeding 1 and can be exhibited as proper decimal fractions. These fractions can be always regarded as interminable for even if terminable they may be rendered interminable by an adjunction of zeros behind



MAPPING A SQUARE ON A LINE

the last significant figure. Let us then write these decimal fractions in the form:

$$x = .\, a_1 \mid a_2 \mid a_3 \mid a_4 \mid a_5 \mid a_6 \mid \ldots$$

$$y = .\, b_1 \mid b_2 \mid b_3 \mid b_4 \mid b_5 \mid b_6 \mid \ldots$$

Now let us form a third decimal fraction z by taking alternately the figures x and y

$$z = .\, a_1 \mid b_1 \mid a_2 \mid b_2 \mid a_3 \mid b_3 \mid a_4 \mid b_4 \mid \ldots$$

This fraction again represents a real number and we can exhibit it as a point Q on the segment OC. The correspondence thus established between P and Q is reciprocal and unique; for given x and y we can always form z, and in only one way; and conversely, the knowledge of z permits us to reconstruct the numbers x and y, and therefore the point P.

What lies between and what lies beyond?

There is nothing in the Cantor theory to preclude the possibility of a transfinite number greater than a, but less than c. Yet all known point-aggregates are either denumerable, like the rational or algebraic number domains, or else, like the transcendentals, have the power of the arithmetic continuum. All attempts to erect an artificial point-aggregate which would be "mightier" than the natural sequence, but less "mighty" than the aggregate of the points on a line, have so far not been crowned with success.

On the other hand, aggregates are known which have a power greater than c. Among these there is the so-called *functional* manifold, i.e., the *totality of all correspondences* which can be established between two continua. This totality cannot be matched with the natural numbers. The corresponding cardinal number is denoted by f. Again, there is nothing in the theory that would preclude the existence of cardinals between c and f, and yet no aggregate has yet been discovered of power less than f, but greater than c.

And beyond f there are still greater cardinal numbers. The same diagonal procedure which permits us to derive the functional "space" from the continuum can be used to derive from the functional space a *superfunctional* which *cannot* be matched with the aggregate of correspondences. Aggregates of higher and

higher power can be thus erected, and the process *cannot conceivably be terminated.*

So pushed to its ultimate border, the Cantor theory asserts that there is *no last transfinite number.* This assertion is strangely similar to the other: there is *no last finite number.* Yet the latter was admittedly an assumption, the fundamental assumption of *finite* arithmetic, whereas the analogous statement in the arithmetic of the *infinite* seems to be the logical conclusion of the whole theory.

There is no last transfinite! The proposition sounds innocent enough, and yet it contained within itself an explosive which nearly wrecked the whole theory, and that at a time when Cantor, having overcome the powerful resistance of his first opponents, had all reason to believe that his principles had emerged triumphant. For almost simultaneously a series of "phenomena" were uncovered which, while seemingly different in character, indicated that something was wrong. The Italian Burali-Forti, the Englishman Bertrand Russell, the German König, and the Frenchman Richard unearthed antinomies and paradoxes, which bear their respective authors' names. Again, the question was raised as to the validity of the Cantor methods and deductions, as to the legitimacy of the use of the actually infinite in mathematics.

It would take me too far to go into detail about the nature of the contradictions discovered. Heterogeneous though these paradoxes are, they all seem to hinge on the questions how the word *all* should be used in mathematics, if it is to be used at all. If this word can be used freely in connection with *any* conceivable acts of the mind, then we can speak of the *aggregate of all aggregates.* If now this is an aggregate in the Cantor sense, then

it too must possess a cardinal number. This transfinite number is the "greatest conceivable," for can we conceive of an aggregate mightier than the aggregate of all aggregates? This cardinal is, therefore, the *last transfinite,* the truly ultra-ultimate step in the evolution of the abstraction which we call number! And yet there is not last transfinite!

Much water has passed under the bridge since these paradoxical questions were first raised; many solutions have been offered, thousands of essays have been written on both sides of the question, much sarcasm indulged in by the Cantorians and their opponents. Yet the question remains wide open. Cantor found mathematics undivided; he left it split into two contending camps.

To present the "platforms" of these opposing mathematical "parties" with the simple means at my disposal, and pledged as I am to avoid technicalities, is impossible. And yet I should fall short of my own program were I to pass up entirely this most vital issue of modern mathematics. So I shall simply and briefly state the dilemma as voiced by the chief representatives of the opposing camps.

On the side of the "formalists" are Hilbert, Russell, Zermelo. While defending Cantor they are "mensheviki" in the sense that they are attempting save his *minimum* program. They admit that the unrestricted use of the words "all," "aggregate," "correspondence," and "number," is inadmissible. But the solution lies not in the complete negation of the theory of aggregates but in the remolding of the theory along the lines of pure reason. We must devise a body of axioms which could serve as the basis of the theory and, to make sure that we are not again led astray by our institution, we must construct a purely *formal,* logically consistent

schematic outline of such a body, a mere skeleton without content. Having erected such a comprehensive, consistent system, we shall base the arithmetic of the infinite upon it as a foundation, secure in our conviction that no paradox or antimony will ever arise again to disturb our peace of mind. Says Hilbert: "From the paradise created for us by Cantor, no one will drive us out."

The *intuitionists*, beginning with Kronecker and reinforced by Poincaré, who in our own day are represented by such great minds as Brouwer in Holland, Weyl in Germany, and to a certain extent by Borel in France, have a different story to tell the definition of an aggregate. The disease antedates Cantor, it is deep-seated, and the whole body mathematical is affected. Says Weyl:

> "We must learn a new modesty. We have stormed the heavens, but succeeded only in building fog upon fog, a mist which will not support anybody who earnestly desires to stand upon it. What is valid seems so insignificant that it may be seriously doubted whether analysis is at all possible."

To the intuitionists the issues go far beyond the confines of the theory of aggregates. They maintain that in order that a concept may gain admission into the realm of mathematics, it is not enough that it be "well defined": it must be *constructible*. Not merely must the concept exist in name, but also an actual construction should be given to determine the object which the concept represents. As to construction, the only admissible ones are the finite processes, or—and this is, indeed, a compromise—such infinite processes as are reducible to finite by means of a finite number of rules. The act of conceiving simultaneously an

infinite number of single objects and of treating the whole as an individual object does not belong to this category of admissible concepts and must *a priori* be barred from arithmetic. And not only does this mean the scrapping of the theory of aggregates, but even the concept of irrational numbers must undergo a profound modification until analysis is purged of all the impurities with which the indiscriminate use of the infinite has polluted it. "For," says Weyl, "mathematics, even to the logical forms in which it moves, is entirely dependent on the concept of natural number."

While this conflict as to the validity of the foundation on which analysis rests is in full blast, the structure itself is rising at a prodigious rate. Each year sees advances which in the nineteenth century would have required the work of decades. Every decade witnesses the opening of new fields of inductive knowledge which voluntarily submits to the penetration of mathematical analysis. And as to physics, which was among the first conquests of analysis, the pancosmos of the Relativity theory is but a universe of differential forms, and the discontinuous phenomena of the microcosmos seem to obey the laws of a wave mechanics which to all appearances is just an application of a theory in differential equations.

And we see that strange spectacle of men who are loudest in proclaiming that the empire rests on insecure foundations—we see these gloomy deans forsaking from time to time their own counsels of alarum to join in the feverish activity of extending the empire, of pushing further and further the far-flung battle-line.

Of such is the kingdom of logic!

From the day on which man miraculously conceived that a brace of pheasants and a couple of days were both instances of the number two, to this day, when man has attempted to express in numbers his own power of abstraction—it was a long, laborious road, and many were the twists and turns.

Have we reached an impasse? Must we retrace our steps? Or is the present crisis just another of these sharp turns from which, if the future be judged by the past, number will again emerge triumphant, ready to climb to still dizzier heights of abstraction?